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Characterization of Inductive Reasoning in Middle School Mathematics Teachers in a Generalization Task

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ABSTRACT

This paper reports on the characterization of the inductive reasoning used by middle school mathematics teachers to solve a task of generalization of a quadratic pattern. The data was collected from individual interviews and the written answers to the generalization task. The analysis was based on the representations and justifications used to move from particular cases to the formulation of the general rule. Three processes characterize the inductive reasoning of the mathematics teachers to obtain a general rule: observation of regularities, establishment of a pattern and formulation of a generalization; while some teachers revealed problems in moving from the observation of regularities to the formulation of a generalization. Therefore, some difficulties in generalizing associated with these processes are also mentioned.

Keywords: inductive processes, task, mathematical generalization, teachers

INTRODUCTION

Inductive reasoning plays a significant role in enhancing intellectual development processes such as intelligence and reasoning strategies (Hayes, Heit, & Swendsen, 2010; Klauer & Phye, 2008; Klauer, Willmes, & Phye, 2002; Mousa, 2017). It has been highly regarded as a way to develop students' abilities to solve problems and to generalize different mathematical patterns (Cañadas, Castro & Castro, 2008; Haverty, Koedinger, Klahr, & Alibali, 2000; Murawska & Zollman, 2015; Neubert & Binko, 1992; Sriraman & Adrian, 2004). In the same sense, the National Council of Teachers of Mathematics [NCTM] (2000) stresses the need to develop middle school students' proficiency in using inductive (and deductive) reasoning to examine structures, develop arguments and formulate generalizations about linear or quadratic patterns. To a great extent, this is due to the fact that inductive reasoning allows the discovery of invariant characteristics between particular cases and to synthesize these characteristics in a general rule (Bills & Rowland, 1999).

For these reasons, for some time, several studies in the field of mathematics education have been based on the interest of the analysis of students' inductive reasoning and how to develop it in a school setting (Molnár, 2011; Papageorgiou, 2009; Sriraman & Adrian, 2004; Tomic, 1995). Some researchers have focused on elementary school students' cognitive processes in discovering regularities (Christou & Papageorgiou, 2007; Papageorgiou, 2009); describing the inductive reasoning's stages and strategies used by secondary school students when solving problems (Cañadas & Castro, 2007; Cañadas, Castro, & Castro, 2008; Cañadas, Castro, & Castro, 2009), and identifying undergraduate students' inductive processes to generalize functional relationships (Haverty et al., 2000).

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Developing inductive reasoning is not only a matter for students but also for teachers. The teacher has the important task of advancing and looking out for the inductive reasoning of the students, as well as helping them to recognize its limitations and possibilities (NCTM, 2000). In addition, teachers need the ability to identify abductive and inductive actions in order to notice and explain students' actions when generalizing from particular instances (El Mouhayar & Jurdak, 2013; Rivera & Becker, 2007). In this regard, previous research has shown that teaching and noticing forms of mathematical reasoning, including inductive, has been cognitively challenging for teachers (Callejo & Zapatera, 2017; Herbert, Vale, Bragg, Loong, & Widjaja, 2015; Melhuish, Thanheiser, & Guyot, 2018; Stylianides, Stylianides, & Traina, 2013). For example, El Mouhayar (2018) points out that recognizing properties and relationships of general aspects of a pattern in far generalization tasks is complex for them.

Generalizing mathematical patterns requires perceiving and developing mathematical structures from particular cases (Clements & Sarama, 2009; Rivera, 2010; Rivera & Becker, 2016). In this study, we considered that the teachers' ability to reason inductively is a factor that influences the ways of advancing and attending to students' reasoning actions to generalize. From a cognitive approach, inductive reasoning helps connect particular instances and systematize pattern detection leading to generalizations (Cañadas & Castro, 2007; Molnár, Greiff, & Csapó, 2013). We examine the inductive reasoning of middle school teachers when generalizing. Clarifying what inductive processes teachers use and how they connect these processes to obtain generalizations would allow teachers and researchers to generate strategies to identify and improve them.

RESEARCH ON TEACHERS' INDUCTIVE REASONING

Some studies about teachers' inductive reasoning have been carried out with pre-service teachers (Manfreda, Slapar, & Hodnik, 2012; Rivera & Becker, 2003, 2007). They have used generalization tasks of linear and quadratic patterns represented in a numerical and figural way. Arslan, Göcmencelebi and Tapan (2009) claim that the pre-service teachers' answers to mathematical problems are essentially based on inductive rather than deductive arguments. Research indicates that pre-service teachers induce general rules of a quadratic pattern using different strategies. For example, most of them use numerical similarity strategies over figural ones when generalizing in an inductive way, but who induces by figural similarity seems to be more capable of justifying and making sense of their generalizations than the others (Rivera & Becker, 2003, 2007). Manfreda et al. (2012) argued that all strategies are not equally effective in generalizing a quadratic pattern. They concluded that product, binomial and sum strategies were more effective for making generalizations by quadratic function, whereas the difference strategy (recursive) gave the least correct generalizations.

On the other hand, several investigations highlighted that pre-service elementary and middle school teachers face difficulties in generalizing from a finite class of particular cases; the greater difficulties are found in the non-linear patterns such as quadratic ones (e.g. Alajmi, 2016; Hallagan, Rule & Carlson, 2009; Rivera & Becker, 2003). Research on inductive reasoning reports that these difficulties are linked to their different skills and ways of generalizing quadratic patterns. For example, Manfreda et al. (2012) found different levels of depth in solving generalization tasks by pre-service teachers; although most of them identify the numerical pattern, few are able to form the general expression. They tend to focus on invariant attributes in numbers over relationships (Rivera y Becker, 2003). The most frequently used strategy to generalize quadratic patterns is the recursive one, but this strategy makes it more difficult to recognize the structure of the quadratic pattern and reach the generalization (Alajmi, 2016; Manfreda et al., 2012; Yeşildere & Akkoç, 2010).

Despite these difficulties, the study of the processes underlying this form of reasoning in middle school teachers has not been discussed in depth in the literature. Therefore, the aim of this paper was to characterize the cognitive processes involved in inductive reasoning that enable teachers to produce correct generalizations, specifically the general rule of a quadratic pattern. With this goal in mind, the research question to answer was: Which are the processes of inductive reasoning used by middle school teachers for obtaining the general rule of a quadratic pattern?

The current study contributes to the literature on this subject in two aspects. First, we offer a characterization of in-service teachers' inductive processes that enable them to reach the successful completion of quadratic pattern generalizations. Second, while we focus on the teachers' inductive processes, we also identify the types of difficulties in generalizing related to such processes.

THEORETICAL FRAMEWORK

Inductive Reasoning

There are several conceptualizations used to analyze inductive reasoning; for example, Klauer (1990) understands it, from a psychological approach, as a mental process that compares attributes of objects and relationships between objects to discover regularities. In turn, Pólya (1957) characterized it, from the field of mathematics, as a research method of the scientific knowledge and the resolution of problems. He established it as "the process of discovering general laws by the observation and combination of particular instances" (p. 114).

In this sense, it is widely accepted that inductive reasoning facilitates noticing patterns to go from the particular to the general; therefore, it is useful whenever it is necessary to discover something general or a regularity (Klauer, 1990). In this way, a large number of studies have examined the inductive reasoning of students and pre-service teachers when solving problems (e.g., Cañadas, Castro, & Castro, 2009; Christou & Papageorgiou, 2007; Haverty et al., 2000; Manfreda et al., 2012; Rivera & Becker, 2003), and they have used mathematical generalization tasks.

According to Dreyfus (2002), to generalize is "to derive or induce from particulars, to identify commonalities, to expand domains of validity" (p. 35). So, to solve a generalization task, individuals need to move from particular cases to the general case (i.e. reason inductively). Such studies coincide in pointing out that inductive reasoning and generalization are two connected cognitive processes. Therefore, this form of reasoning could be promoted with tasks that require induction or obtaining the general rule that governs a set of specific elements (Glaser & Pellegrino, 1982). Klauer (1996) interprets this connection as "the end product of an inductive reasoning process is the discovery of a generalization or the disproving of an assumed generalization" (p. 38).

The most common tasks are the figural patterns that are *high in Gestalt goodness*, this is, the pattern "tends to have an interpreted structure that reflects the orderly, balanced, and harmonious form of the pattern" (Rivera, 2010, p. 304). This means that the configuration of the particular instances of the pattern made it possible to appreciate similitudes between them and notice the structure of the pattern. In consequence, it orients and makes the construction of the algebraic rule of the pattern easier. However, Rivera and Becker (2003) claim that pre-service teachers have more difficulties in making inductions when the tasks contain homogeneous figures and numbers of higher order (or, in general, specific objects in a category) that possess few common properties and more perceptual dissimilarity than similiraty. Acording to the definition of Rivera (2010), these types of tasks can be considered with patterns that are *low in Gestalt goodness*, because they have a more complex interpretative structure with parts that are not easily discernable.

In consequence, assuming inductive reasoning as a mental process oriented to infer laws or general conclusions through the observation and connection of particular instances of a class of objects or situations, we are interested in the examination of the cognitive processes involved in the inductive reasoning that allows in-service mathematics teachers to make successful generalizations. Our particular interest is focused on tasks whose structure is not easily discernible and that need a deeper conceptualization to discover and generalize the pattern.

This interest is connected to the lack of empirical information about the inductive reasoning of in-service teachers; most of the studies have been focused on students (Cañadas & Castro, 2007; Christou & Papageorgiou, 2007; Haverty et al., 2000; Reid, 2002). Therefore, we consider that the recognition of these processes and possible difficulties could help in understanding why and how the teachers can or cannot produce generalizations.

Processes of Inductive Reasoning

As previously stated, we assume that inductive reasoning is a means or route to make generalizations from particular cases, because this type of generalization implies the identification of an invariant property or characteristic in a class of objects or specific cases and takes it into the general case (Bills & Rowland, 1999; Davydov, 2008). A general rule that describes the invariant characteristic in all cases of a class is a way to express a mathematical generalization.

In this sense, Pólya (1967) proposed four phases of inductive reasoning to figure out principles, properties and general rules, these are: (1) look for some similarity in the particular cases; (2) conjecture formulation about this similitude; (3) make a generalization and (4) verify the generalization testing with new cases. This author claims that the identification of the pattern associated to common characteristics and variations between the cases is a fundamental process to achieve a proper generalization. In a similar way, Reid and Knipping (2010) emphasize the observation and prediction of the pattern, conjecturing, generalizing and testing as distinctive actions of inductive reasoning. These authors point out that, cognitively speaking, the prediction of a pattern leads to a conclusion that could be a conjecture or a generalization depending on whether an additional verification is required or not.

The phases proposed by Pólya to achieve mathematical generalizations involve the use and connection between three cognitive processes underlying the inductive reasoning: observation of a regularity, establishment of a pattern and formulation of a generalization, these processes are related with the first three phases mentioned above. The fourth phase, verification of the generalization does not match with the development of the inductive reasoning, but with a proof. The general rule or conclusion of the inductive process could be validated with an empirical argumentation, or with a deductive proof (Davydov, 1990; Cañadas, Castro, & Castro, 2009).

So, based on the literature, we will describe these processes in order to understand the inductive reasoning of mathematics teachers to obtain a general rule and how these processes are interconnected.

a. Observation of a regularity

Observation of a regularity is understood as a cognitive process based on the mental action of comparing with the aim to identify similitudes, differences or what remains invariant in a set of objects or particular cases (Klauer, 1990). This process goes with phase (1) of Pólya's method because inductive reasoning starts with the analysis of particular cases of a situation and is aimed at the observation of some regularity. Klauer and Phye (1994) argued that regularities are essential to establish categories and discover a general rule.

b. Establishment of a pattern

Establishment of a pattern is an essential cognitive process to determine a general rule from particular cases (Haverty et al, 2000). The pattern represents what is repeated regularly in a set of specific cases or facts (Castro, Cañadas & Molina, 2010). This process goes with the phase (2) mentioned by Pólya because conjecture is a proposition on mathematical patterns and its application to other possible cases of the situation, even unknown cases. Therefore, in order to formulate a conjecture, it is necessary to identify a possible pattern in the observed cases and to express it in a verbal, numerical or other form of mathematical representations (Cañadas & Castro, 2007).

According to Clements and Sarama (2009), "identifying and applying patterns help bring order, cohesion, and predictability to seemingly unorganized situations and allows you to make generalizations beyond the information in front of you" (p. 190). For these authors, the establishment of a pattern implies connecting regularities and mathematical structures. So, identifying what is repeated in a regular way is necessary to establish a pattern, but it is not enough. The identification of the mathematical relationships underlying the regularity is also needed; these relationships in the pattern could be numerical, spatial or logical (Mulligan & Mitchelmore, 2009).

c. Formulation of a generalization

Pólya (1967) proposed the formulation of a generalization as phase (3); this means moving from the consideration of a given set of objects to a larger set that contains the first one. In a cognitive sense, formulation of a generalization is a process that requires the abstraction of the general aspect of a set of particular instances by connecting the individual phenomenon within a certain whole (Davydov, 1990). This process consists of an extension of the pattern to all cases (Cañadas & Castro, 2007); therefore, it helps to move from the pattern to the expression of the general rule.

Additionally, the generalization of a mathematical pattern involves using and describing structural relationships or structures (Küchemann & Hoyles, 2009), such as describing relationships between variables using linear, quadratic, or any other mathematical structure.

METHODOLOGY

Participants and Context

This study was carried out with nineteen middle school mathematics teachers (nine women and ten men) with between five and 20 years of teaching experience in public schools in Mexico. They were invited to participate as part of their teacher professional development in mathematics didactics, without receiving any remuneration for this and without the results influencing their work position. These teachers agreed to participate voluntarily in this study.

The participating teachers had professional qualifications as middle school teachers or engineers, so they all studied the mathematical concepts such as sequences, functions, and linear and quadratic equations. The selection criterion of the participants was to have at least one year of experience teaching quadratic sequences using numerical and figural patterns.

Design of the Instrument

Obtaining a general rule of quadratic patterns is one learning objective of the mathematical curriculum in middle school in Mexico (SEP, 2011). This is connected with the teaching and learning of sequences and quadratic equations in the third year (9th grade; ages between 12 and 14 years old) in this educational level, through the use of numerical and figurative pattern tasks in official textbooks.

Taking into account the above, two tasks of inductive reasoning on quadratic sequences were design and implemented (Glaser & Pellegrino, 1982). These tasks required the general rule of the quadratic behavior of one variable based on the representation of a set of particular cases. So, the general rule corresponded with the nth term of a second grade sequence.

The particular cases in each task could be expressed verbally, numerically or geometrically and the rule could be expressed verbally or algebraically. This means that it was possible to use different systems of representations to solve the tasks and the solution was not unique, so the strategies used to go from particular cases to the general rule could differ or even be complementary (Cañadas, Castro & Castro, 2009); for example, a strategy based on arithmetical calculations and another on the visual analysis of particular cases.

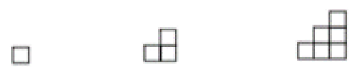
The tasks involved generalizing patterns that are low in Gestalt goodness, so that the structure of the pattern was not clearly displayed in the representation of the particular instances. Also, to induce the general rule required to rely on work with numerical relationships and structures. In this regard, based on study by Gentner (as cited on Rivera & Becker, 2003), we assume that older children and adults are able to perform similarity based on relations that structure the objects rather than attributes observed in objects.

In this paper, the results of the investigation are reported and discussed based on the data of the first task, "the toothpicks", because it is the one where most of the teachers achieved the generalization and the data allowed a more solid argumentation of the results.

The toothpicks task

This task showed a sequence of figures with the form of stairs of one, two and three floors formed by toothpicks (**Figure 1**). The task was to induce a rule to determine the number of floors (*n*) that could be built with a specific number of toothpicks (*c*). This task is adapted from an inductive reasoning task proposed by Cañadas, Castro and Castro (2009, p. 270) that asked for the number of toothpicks for the fourth, fifth and sixth stair. In this study, the participants were asked for the number of floors with 180 toothpicks that corresponds with a far term in the sequence in order to formulate a general rule and avoid to recursive reasoning.

The next figures represent stairs of one, two and three floors build with toothpicks.



Determine how many floors should have a stair with 180 toothpicks.

Figure 1. Task of the tool used to collect data

The task demands examining the associated pattern in each figure and determine a functional relation between the number of toothpicks in each figure and its position. A quadratic expression could be used to describe this relation. Another way to proceed is to consider the number of floors as a variable instead of the position of the figure. Specific values of n and c are particular cases in the task; the general rule could be obtained by connecting the following inductive processes:

a. **Observation of regularity.** A possible regularity to observe in the variation of the sequence of figures is the number of toothpicks in each stair varies in a quadratic way as the position of the figures increases. This regularity could be observed and confirmed with the finite differences strategy or a visual analysis of the figures focusing on the number of toothpicks.

b. **Establishment of a pattern.** A verbal way to express the pattern is to establish that the number of toothpicks in a figure corresponds with the square of the value of its position plus the value of this position multiplied by three. This pattern could also be expressed by a numerical arrangement as shown below:

$$\begin{array}{cc}
n & c \\
1 & 4 = 1 + 3(1) \\
2 & 10 = 4 + 3(2) \\
3 & 18 = 9 + 3(3) \\
\vdots & \vdots
\end{array}$$

This type of numerical representation of the pattern is not unique; it could also be represented by the following mathematical structure:

$$\begin{array}{cc}
n & c \\
1 & 4 = 1(1 + 3) \\
2 & 10 = 2(2 + 3) \\
3 & 18 = 3(3 + 3) \\
\vdots & \vdots
\end{array}$$

c. **Formulation of the generalization.** The general rule could be expressed algebraically in any of the following forms according to the underlying structure of the pattern: $c = n(n + 3)$; $c = n^2 + 3n$; $c = 4n + (n - 1)n$.

Data Collection

The teachers answered both tasks individually in periods of between 20 to 30 minutes per task. The information was collected from the answer sheets of each teacher. The teachers were asked to show and justify their complete procedure of solution in each task.

After the analysis of the answer sheets, the participants were interviewed with the intention of understanding in greater depth their inductive reasoning process, from working with particular cases to generalization, and dialoguing about the difficulties faced to obtain the general rule. In the interview, they were asked to explain the reasoning used to solve the tasks in an oral form. Finally, they were asked to provide reasons for propositions or decisions insufficiently argued in their answers. All information was registered in audio recordings. The application of the tasks and the interview was carried out by the main author of this article, and complemented with the notes taken by the research team.

Data Analysis

It is said that reasoning emerges or is produced when a person transforms representations of the information of a situation in other forms of representation to express a conclusion (Leighton & Sternberg, 2004). In fact, the mathematical reasoning needs mental, verbal, visual or other representations for its development and communication (Dreyfus, Nardi & Leikin, 2012; Hiebert & Carpenter, 1992). Therefore, the representations used by the participants to express their reasoning were analyzed under this consideration in order to identify and describe the inductive processes of the teachers. In this sense, it is worth reiterating that both, the particular cases and the general rule could be expressed or represented in numerical, geometrical, verbal or algebraic forms in these tasks.

The answers of the participants were organized and analyzed in two stages. The answers that included a verbal or algebraic expression of a generalization were selected in the first stage. The answers were organized into two categories, C1 and C2, according to whether the general rule was obtained or not in each task, respectively.

In the second stage, in order to identify similarities and differences between the reasoning processes that allowed teachers to reach generalization, the processes present in the answers of category C1 were analyzed and compared. Subsequently, the processes identified in C1 were contrasted with the answers of category C2 to identify the types of difficulties related to such processes. The analysis and categorization of the inductive processes as well as the difficulties identified were triangulated with the research team for its delimitation and argumentation.

The participants were assigned pseudonyms using the English alphabet, instead of names, to respect their anonymity: Teacher A, Teacher B, etc. The distribution of the answers provided by the teachers in each category is:

C1: The generalization is achieved (A, B, C, D, E, F, G, H, I, J and K)

C2: The generalization is not achieved (L, M, N, O, P, Q, R and S).

RESULTS

Teachers' Inductive Processes to Obtain a General Rule

The inductive reasoning of the mathematics teachers was studied based on the answers and justifications given during the task that required the induction of a general rule of a quadratic pattern. It was identified that 11 of the 19 participants managed to obtain the general rule by reasoning inductively. This form or reasoning was characterized by following three processes: *observing regularity, establishing a pattern* and *formulating a generalization*. The remaining eight teachers experienced difficulty moving through the three processes observed in their other eleven colleagues. The following sections describe and provide evidence of the three processes used by the teachers that managed to obtain the general rule (teachers in category C1).

Observation of regularities

The initial process of the teachers that managed to obtain a generalization was the observation of regularities. This process consisted in the comparison and search for relationships between the particular cases based on their invariant and dissimilar characteristics. Therefore, the establishment of a pattern was based on the observation of regularities on a local and global level. The *local observation* took shape with the identification of a relationship between a particular case and the next one; while the *global observation* took shape with the identification of a common characteristic to every particular case and the relationships between them.

Even though all teachers observed regularities, only eleven observed regularity on a global level; six of them made a numerical analysis and five of them supported in a visual way. That being so, the arithmetical calculations and the visual perception were the central elements in the process of observation of regularities.

Observation of regularities in a numerical way

Observation of regularities in a numerical way started with obtaining some particular cases of the given figures. The teachers counted each case: 4, 10 and 18 toothpicks in the stairs of 1, 2 and 3 floors, respectively (**Figure 2**). Then, they organized these numerical data in rows or columns, as in a table. The geometrical representation (figural) of the new particular cases, such as the figure of four floors, helped in the observation and verification of regularities.

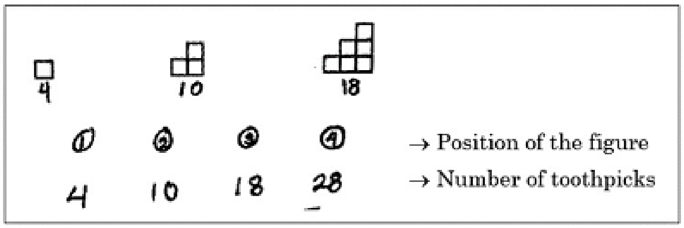


Figure 2. Recognition and organization of particular cases by teacher A

The regularities observed and processed numerically were expressed in the following way:

i. The number of toothpicks increases from one stair to another with a non-constant variation, and
ii. The second differences of these numbers are always equal to the value of 2.

The most used strategy to observe these regularities was the recursive algorithm of the differences between the number of toothpicks in each figure and the previous one. For example, Teacher A used the first differences between the number of toothpicks that formed the stairs with 1, 2, 3, 4 steps, etc. to calculate the number of toothpicks in the fourth, fifth, sixth position and so on (**Figure 3**). This is, $10 - 4 = 6$, $18 - 10 = 8$ and $28 - 18 = 10$. Then, the teacher obtained the second differences and determined that there was a constant value that equals to 2. Later, with inverse operations (additions), the teacher determined the number of toothpicks (40, 54, 70…) that conformed the following stairs in the sequence of figures.

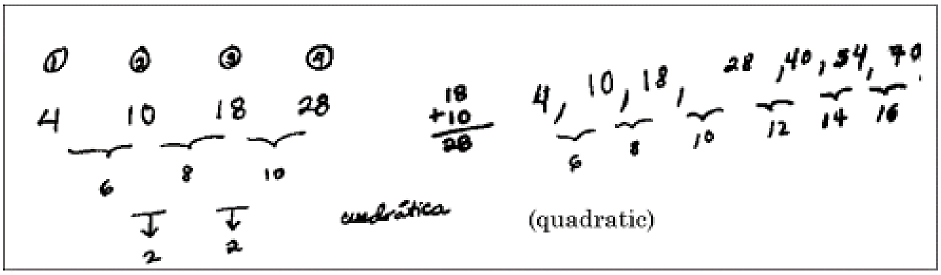


Figure 3. Process to observe local regularities of Teacher A

Observing a global regularity involved the identification of a relationship that described an invariant characteristic between several particular cases. The teachers observed three types of regularities in the global behavior of the number of toothpicks in the figures. They go further than the recursive differences by looking for multiplicative and additive relationships between the numerical values of the first cases and focusing on the analysis of the increasing behavior of the values.

A global regularity observed was that the number of toothpicks in the first four figures could be expressed as a sum of two numbers: the square of a number and a multiple of three. Teacher A, for example, observed this regularity by comparing the number of toothpicks and the position of the figure based on additive relationships. She decomposed the number of toothpicks of the first four stairs (4, 10, 18, 28) as the sum of the square of its position and a number (**Figure 4**). In numerical terms: $4 = 1 + \mathbf{3}$; $10 = 4 + \mathbf{6}$; $18 = 9 + \mathbf{9}$ and $28 = 16 + \mathbf{12}$. She noticed that the second summands in these additive relationships (3, 6, 9 and 12) are multiples of 3.

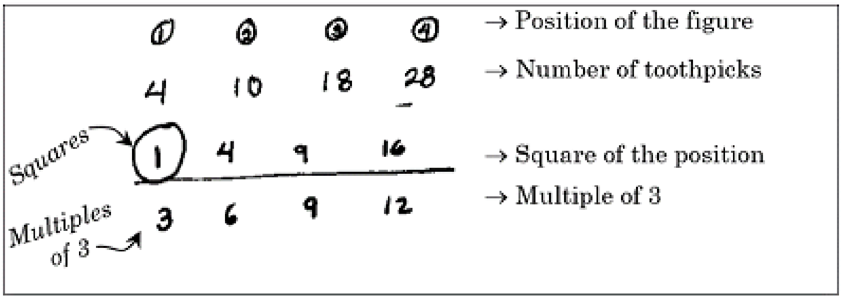


Figure 4. Process to observe a global regularity of teacher A

Another global regularity observed by the teachers to describe the behavior of the number of toothpicks is that, in the first figures, the number of the figure equals the product of two numbers (the number of floors and this same number added by three). Teacher B was looking for a multiplicative relationship between the total number of toothpicks and the number of floors in the first, second and third position in the sequence when he found this regularity (**Figure 5**).

1) I counted the number of toothpicks in the stair of each figure:

1 floor = 4 toothpicks

2 floors = 10 toothpicks

3 floors = 18 toothpicks

2) I looked for any regularity in the total number of toothpicks, I mean between 4, 10 and 18.

3) In this way, I looked for a number that multiplied by the number of floors gave me the number of toothpicks used to build the stair.

4) After I found the numbers, the multiplications were:

(1 floor) 1 piso $\rightarrow 1 \times 4$

(2 floors) 2 pisos $\rightarrow 2 \times 5$

(3 floors) 3 pisos $\rightarrow 3 \times 6$

Figure 5. Process to observe global regularities of Teacher B

Teacher B described his initial reasoning to solve the task as:

Teacher B: The first thing I did was to observe the cases presented in the problem, the particular cases… I thought that there should be a regularity between the total number of toothpicks as the number of floors was increasing… Having in mind that the unknown value of the problem is the number of floors in the stair or the number of floors that could be built with 180 toothpicks, then I thought that I should use the number of floors in the stair to find some regularity in the total number of toothpicks. So, I went case by case; in the first case, one floor, I tried using arithmetical operations with any other number to get the total number of toothpicks… I said to myself I am going to use multiplication and using the number of floors, which is one, I looked for a number that gave me four, it was four. Then I did the same for the second case, I looked for a number that multiplied by two gave me 10, it was five. I did the same for the third case, a number that multiplied by three, the number of floors, gave me 18 as a result, it was six. Then I wrote and saw that 1×4, 2×5 and 3×6 gave me the results of the total number of toothpicks.

Researcher: Why did you think of regularity since the beginning?

Teacher B: It was because of the increment in the structure of the figures, the stairs. I saw it was recursive in some sense; I mean the first figure was (contained) in the second figure; the second figure was in the third one and then I was sure that this should be happening in the next figures. If what I wanted to find was the number of floors… the first thing I thought was to try to find the number of toothpicks with some arithmetic operation with the number of floors, such as addition or multiplication; because if I use the number of floors of the stair to find the total number of toothpicks, then I could use multiplication.

Observation of regularities in a visual way

The teachers that observed regularities with visual support identified invariant characteristics in the figures; for example, perceiving similitudes or differences between one figure and the next one. In this sense, they estimated the number of toothpicks in each figure based on their position in the sequence. Locally, they observed that the number of the position in each stair in the sequence is equals the number of floors and also the number of new squares added each time matches the position number of that figure; this is the case of Teacher C (**Figure 6**).

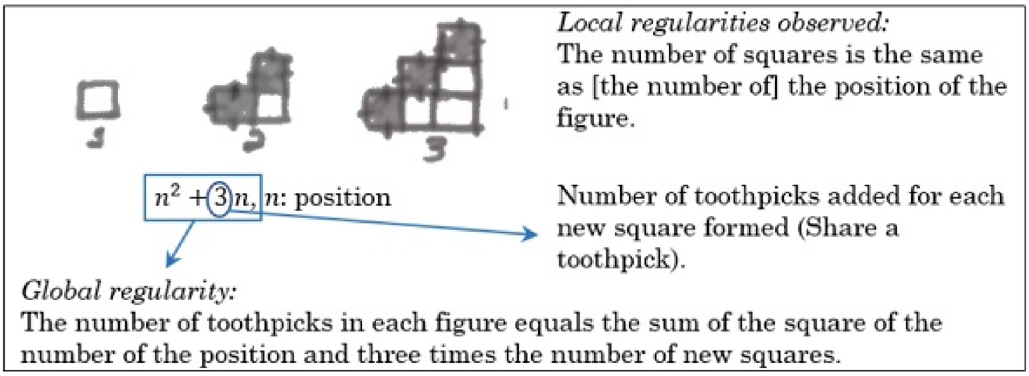


Figure 6. Process of visual observation of regularities of Teacher C

Another invariant characteristic observed by this teacher is that there were three more toothpicks for each new square in the figure of three floors, which he generalized incorrectly. He did not notice that in his count he duplicates some toothpicks belonging to the previous figure, so that the sense given to number 3 in the expression $n^2 + 3n$ is incorrect. However, he continued to analyze the figures paying attention to the shared characteristics between the figures and not on the number of toothpicks added in each. In doing so, he observed a global regularity and rectified the meaning given to the terms in his general expression.

Then, the global observation of regularity consisted of the association of all the particular cases based on these characteristics. This observation is shaped with the next relationship: the number of toothpicks in each figure equals the sum of the square of the number of the position plus three times that number. This is a global regularity because it establishes a relationship between all particular cases based on a shared invariant characteristic.

Establishment of a quadratic pattern

This study identified that the teachers established a pattern by recognizing and associating the observed regularities with a structure that describes and norms the global behavior of the particular cases. The teachers established the pattern corresponding with the increase in the number of toothpicks by associating and describing the observed numerical regularity with a quadratic structure. This was a consequence of identifying that the second differences of the number of toothpicks in each figure were always equal to the constant value of two. Accordingly, they recognized that the values of the number of toothpicks match with a quadratic sequence or that it could be represented with a quadratic model (**Figure 7**). So, they defined a functional relation between the number of toothpicks and the number of floors (or the number associated to the position of the figure); in this way, they recognized a quadratic pattern and then obtained the general rule of the sequence.

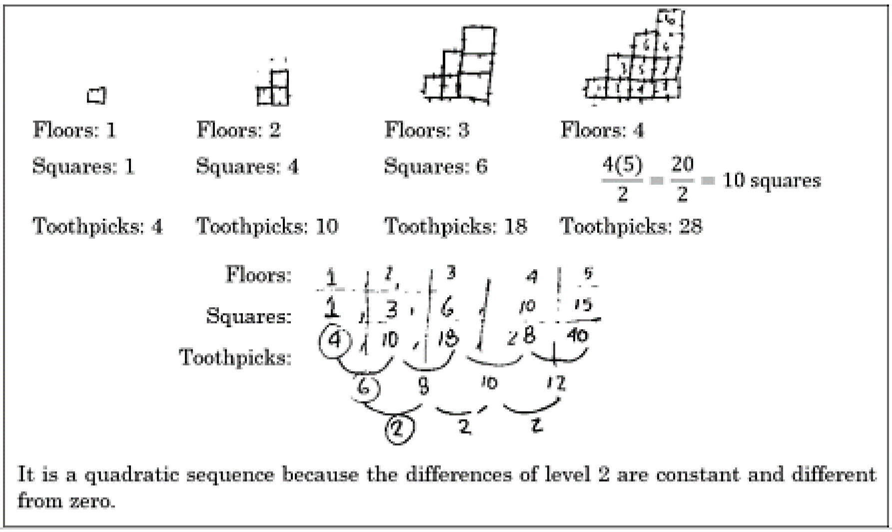


Figure 7. Identification of the quadratic behavior in the data of Teacher D

After recognizing that the number of toothpicks corresponds with the values of a quadratic sequence, the process of establishment of the pattern consisted of connecting the particular cases in a global way through additive and multiplicative relations, and considering if other particular cases that were not in the original sequence of figures satisfied this relation. Three arithmetical structures were detected in the representations used by the teachers to express the quadratic pattern: a multiplicative structure of two linear variable factors, an additive structure formed by the square of a variable quantity and a multiple of this quantity, and an additive structure that combines the multiplicative relations of a variable.

Quadratic pattern expressed as the product of two linear factors

The teachers detected the pattern of the number of toothpicks equals the product of the number of floors and this same number increased by three units. This is a quadratic pattern because it is the product of two quantities with linear variation. The process to establish this quadratic pattern was to look for and use a multiplicative relationship to determine the number of toothpicks in function of the number of floors (**Figure 8**).

(a)

1 piso → 1 × 4
2 pisos → 2 × 5
3 pisos → 3 × 6

So the number of floors is multiplied by the same number but added by three.

(b) Now I found I could use the total number of toothpicks to answer the question, the model for this regularity is:

$$
\begin{gathered}
1 \times (1+3) \\
2 \times (2+3) \\
3 \times (3+3) \\
\vdots \\
n \times (n+3)
\end{gathered}
$$

Figure 8. Process to establish the pattern of Teacher B, based on a multiplicative structure

The teacher used a multiplicative structure to represent the global behavior of the number of toothpicks in different particular cases and establish the pattern. Teacher B describes this:

Teacher B: Then I decided to look for the regularity between the numbers I have already multiplied (**Figure 8-a**) by the number of floors; then the regularity of these numbers is the same number of floors but added by 3 (**Figure 8-b**). For example, the first case was the floor number, which is 1 plus 3 giving 4, which is the number found to multiply by 1 and obtain the number of toothpicks [4]. The second case is the same, the floor number is 2 and added by 3 gives me 5, which is the number used to multiply by the floor number that gave me the number of toothpicks… There is a pattern because I can build its behavior with a rule: To obtain the number of toothpicks I will take the number of floors and multiply this number by this same number added by three.

Quadratic pattern structured by the sum of the square of a number and a multiple

Another way to establish the quadratic pattern was using an additive structure formed by the square of a number and a quantity that varies in direct proportionality. The algebraic model associated to this pattern is of the form: $f(n) = an^2 + bn, a, b, n \in \mathbb{Z}^+$, as Teacher A did. He started looking for numbers such that the sum of this number and the square of the number of the position of the first four figures (position: 1, 2, 3 and 4; square of the position: 1, 4, 9 and 16) equal the number of toothpicks in each figure (4, 10, 18 and 28). The resulting numbers were 3, 6, 9 and 12, respectively (**Figure 9**).

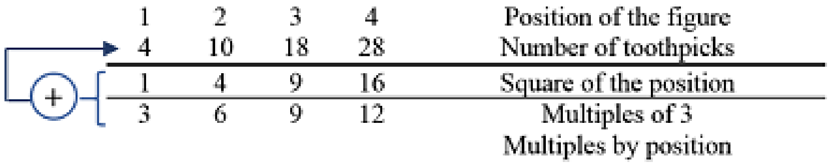


Figure 9. Process of pattern establishment by Teacher A, based on an additive structure

This teacher identified that the numbers obtained (3, 6, 9 and 12) goes up in direct proportion to the number of the position of the figure. Specifically, she established that the numbers are multiples of three of the position. The establishment of the additive and multiplicative relations led her to express that the pattern associated with the number of toothpicks in the stairs equals: the sum of the square of the position of each figure in the sequence and the number of the position multiplied by three.

Quadratic pattern based on the juxtaposition of additive and multiplicative relationships

The quadratic pattern was also established with the juxtaposition of additive and multiplicative relations. For example, Teacher D looked for an additive or multiplicative relation between a particular case and the following one (**Figure 10**). Based on this, he determined that to obtain the number of toothpicks in the figures he should multiply the number of floors in a certain figure [n] and the next one [$n + 1$], and then add the missing number to obtain the total number of toothpicks in the first one. He identified that the missing number was double the number of floors in the first figure [$2n$]. Finally, this pattern was expressed in a general way as: $n(n + 1) + 2n$; this expression matches with the sum of two multiplicative relations in one variable.

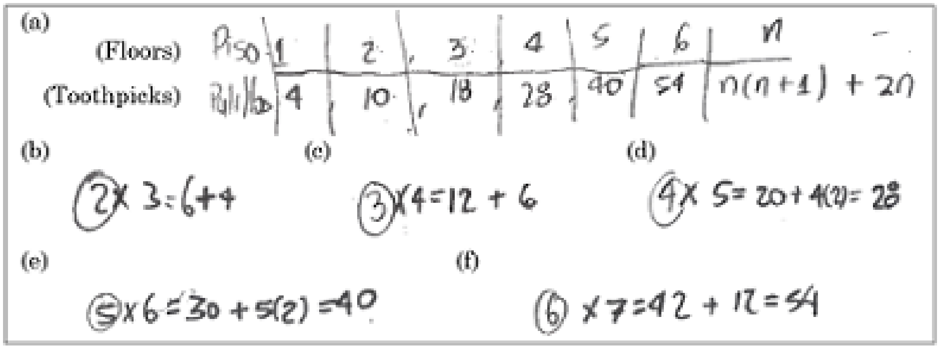


Figure 10. Process of pattern establishment of teacher D based on the juxtaposition of additive and multiplicative relations

Teacher D: From these values (**Figure 10-a**), I was connecting them with arithmetic operations to check if there was any regularity and started doing some math considering the first two cases, the case with one floor and the next one [number of floors: 1 and 2] and the number of toothpicks [4 and 10, respectively], in these four [points out the numbers: 1 and 4, 2 and 10].

Immediately after, Teacher D focused on the values of the figures three and four to describe how he recognized the numerical pattern of toothpicks and said:

Teacher D: Then I chose the number three (**Figure 10-c**), **three** times four is twelve and I just needed to add six, this six was the double of the floor I was choosing [3], and yes, it worked with the rest of the cases. So, I started with **four** and the next one [5], I added one [4+1=5], then four times five is twenty, and I needed 28 [number of toothpicks in the figure with 4 floors] so I noticed that I needed eight which is associated with **the first number** [4], it is the double of four (**Figure 10-d**). And so on with five (**Figure 10-e**) and six (**Figure 10-f**), it worked.

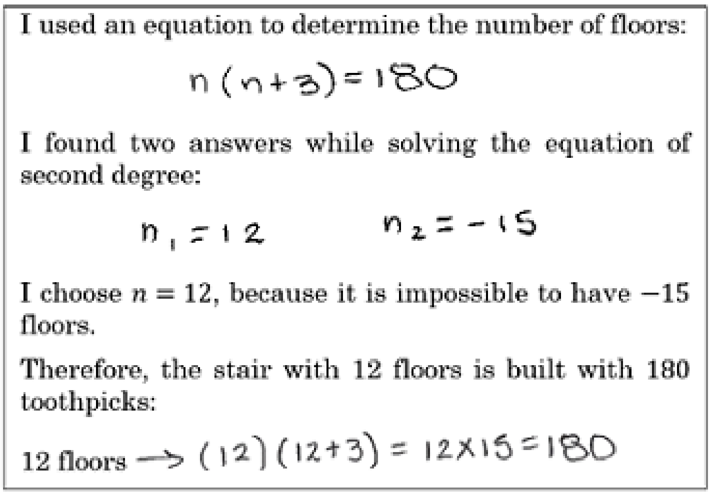
The visual observation of the invariant characteristics in the figures worked, as mentioned before, as a support to associate particular cases and to identify the global behavior of the number of toothpicks. For example, Teacher C (**Figure 6**) considered the connection between the number of toothpicks in each figure and its position by identifying the common characteristics of the form of the figure. Consequently, he established the following pattern: the number of toothpicks in each stair is the result of the sum of the square of the number that indicates its position and this number multiplied by three. The general structure of the quadratic pattern is: $f(n) = n^2 + 3n$; $n \in \mathbb{Z}^+$.

Formulation of a generalization

A common aspect shown by the teachers that made a successful transition from the pattern establishment to the generalization is the ability to take out of context or isolate the pattern of the particular cases analyzed and to expand it to the whole set of cases. The process to formulate a generalization goes beyond the perception of the common aspects between the particular cases to consider the abstraction of an invariant mathematical relation in all the cases of the task. The teachers that achieved the generalization completed the process and obtained the general rule to determine the number of toothpicks of any figure; this rule was expressed in a verbal or algebraic way. They noticed that there was a general way to determine the number of toothpicks in any stair even when the position of the figure or the number of floors varies. For example, Teacher B considered that the number of toothpicks could be obtained in terms of the number of floors (n) and expressed the general rule verbally as:

Teacher B: To find the number of toothpicks, I am going to consider the number of floors; then I am going to multiply this number by this same number added by three.

In order to find the answer of the task, the teacher also expresses the general rule algebraically: $n(n + 3)$, and used this expression to pose and solve the equation $n^2 + 3n = 180$ (**Figure 11**).


I used an equation to determine the number of floors:

$n(n+3)=180$

I found two answers while solving the equation of second degree:

$n_1=12$ $n_2=-15$

I choose $n = 12$, because it is impossible to have −15 floors.

Therefore, the stair with 12 floors is built with 180 toothpicks:

12 floors → $(12)(12+3) = 12\times15=180$

Figure 11. Algebraic expression of the general rule and solution of Teacher B

The teacher described the formulation of the generalization as:

Teacher B: I noticed that the regularity worked in the three cases [the first ones] so I decided to write this regularity in a general way. I named [n] the number of floors and wrote each of them [the number of toothpicks] in terms of n. It turned out that it was n times $n + 3$ in the three cases. Then, the same should happen in the case that the stair had a total of 180 toothpicks and build the equation: n times n plus three, equals to 180 [$n(n + 3) = 180$] and it turns out to be an equation of second degree… I check it [the solution] with the regularity I found in the numbers and it turns out that it was indeed 180.

Two different algebraic expressions for the general rule were also found according to the quadratic patterns established by the teachers. The first expression (**Figure 12**) is based on the number of the position of the figure, and the second one (**Figure 10-a**) is based on the number of floors: $n(n + 1) + 2n$.

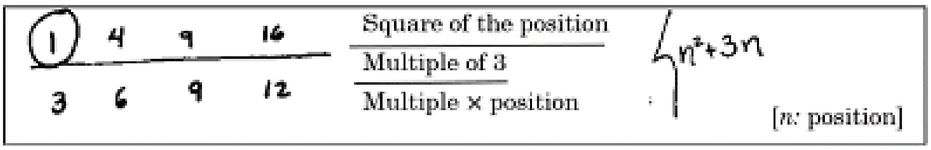


Figure 12. Algebraic expression of the general rule of Teacher A

Another relevant aspect of the formulation of the generalization was extending the pattern to a general category of cases in which the rule obtained was valid. The teachers provide evidence of this aspect when they determine the set of values that the variable n in the algebraic expressions could take. For example, Teacher B established that n belongs to the set of positive integers:

Researcher: What are the possible values of n in this equation?

Teacher B: The values should be in the positive integers because the problem says that the stair should have at least one floor.

Difficulties in Generalizing Associated with Inductive Processes

This study identified that the connection between the three inductive processes mentioned previously makes it possible to generalize a quadratic pattern form specific figural instances. In contrast, the teachers that did not manage to obtain the general rule (Category C2) had cognitive difficulties associated to the process to establish the pattern. In other words, the absence of this process did not allow the interconnection between the observation of regularities and the formulation of a generalization. The description of the difficulties in generalizing associated with the inductive processes are presented in the following paragraphs.

Difficulty in observing a regularity in a global way. Teachers in category C2 showed this difficulty because they could not observe a global regularity that indicates the relationship between the values of a quadratic sequence and restricting the use of the recursive strategy of differences to detect some local regularity in the values. With this process, they were only able to see that the number of toothpicks necessary to build the first steps corresponds with the values of a quadratic function without more specifications, as shown in the work of teacher L (**Figure 13**). He used the following words to explain why it was so complicated to generalize using this strategy:

> Because I could not find a relation. I did not organize the data in a table that could show a better sight than with the differences. Because here [pointing the differences] I was closing myself off looking for an expression to calculate the number of toothpicks based on the number of floors. But I needed to find a relationship... For example, the relation between one and four but through another variable

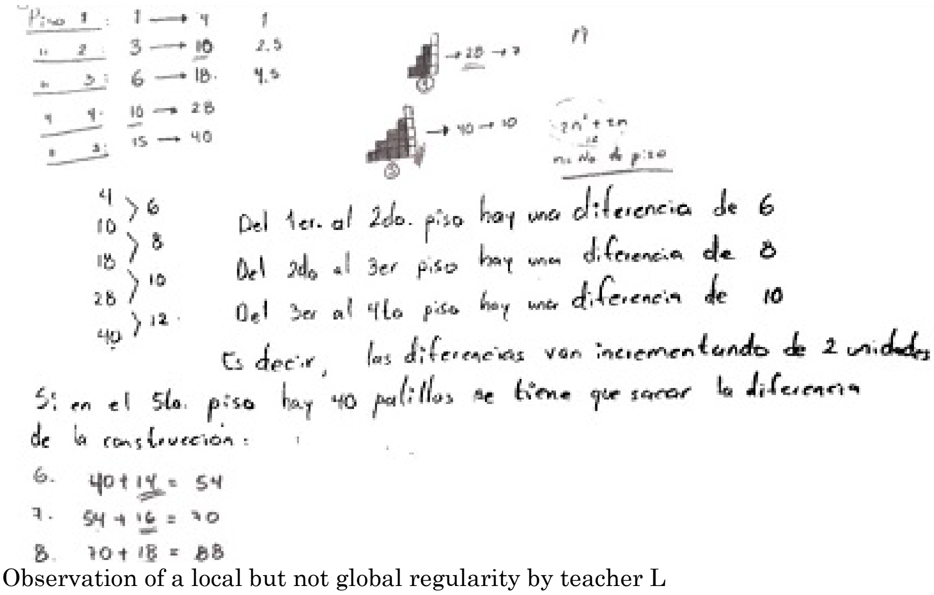


Figure 13. Observation of a local but not global regularity by teacher L

The difficulty detected made it impossible to move on from the work in particular cases to the observation of a global regularity and, therefore, to establish the quadratic pattern. In general, it was found that the establishment of a quadratic pattern based on the numerical work without any visual reference that guides the construction of relations between variables is complex for the teachers.

Difficulty in associating the observed regularities with the mathematical structure. This difficulty was observed in the teachers that could not identify the mathematical structure that describes the quadratic behavior of the values of the dependent variable. They only tried to identify a possible relationship between the values (variables) by trial and error, without the recourse of either additive, multiplicative or both relationships. For example, teacher L wanted to obtain a general formula by considering that the number of toothpicks was associated with the square of a number, adding or reducing some quantity but without any systematic strategy that expressed the observed regularity as a quadratic relation:

> I tried to obtain a general form but it does not fit (no coincidence). I could obtain the total number of toothpicks in some cases, but I get a greater number in others... I should play with these numbers and sometimes with the signs of the numbers [trial and error]; the [second] difference would be an important number but we should remember that we have to add two. For example, in floor 1, let's suppose that n equals 1; if I square this number, I get 1 but adding two does not give me four, it is three [Relation: $n^2 + 2$]. So I have to think of a number, for example, if I multiply the square of 1 by 2 and add the multiplication of 2 by n [$n = 1$], I could get 4 [Relation: $2n^2 + 2n$], n is the number of the floor. But this is in the first floor; in the stair of two floors... the square of 2 is 4, 4 by 2 is 8 plus 2 by 2 gives 12 [the number of toothpicks must be 10 for $n = 2$]. And there it is, I didn't get the result... I squared the number and adding or reducing a difference I obtained the result. But it didn't fit, there should be a general formula to obtain the number of toothpicks needed for any floor.

It was identified that extending what is observed in the particular cases to a set that contains the total of cases leads to the abstraction of the general. This process, together with the knowledge and use of the quadratic structure, are fundamental for the achievement of a general rule of the pattern. In contrast, the lack of abstraction of the general was a factor that prevented the production of the generalization even when some teachers discovered the quadratic pattern. For example, some teachers did not achieve a general expression even when they were able to identify that the number of toothpicks to build a stair with any number of steps could be determined with a formula or a quadratic expression (indicated by the fact that the second differences were constant).

DISCUSSION AND CONCLUSION

This study offers a characterization of three processes underlying the inductive reasoning of mathematics teachers while solving a generalization task of a quadratic pattern. An adequate connection of these three processes —*observation of regularities, establishment of a pattern and formulation of a generalization*— allowed the teachers to establish the general rule associated to the quadratic pattern.

The reasoning used to induce the general rule of a quadratic pattern started with the process of observing local and global regularities in particular cases. This process was supported by the comparison strategy (Klauer, 1990) and the search for relationships between some particular cases. The focus on the observation of local regularities was in the identification of some relationship between a case and the previous or the next one, while the focus of global regularities was in the common relationships between several particular cases. This process was mainly based on arithmetic and visual treatment of the invariant characteristics of the figures.

The identification of arithmetic relationships between the values of a set of specific cases promoted the transition from the observation of regularities to the establishment of a pattern; this enables the recognition of the type of mathematical structure underlying the pattern. As such, the teachers who identify the quadratic patterns are the ones who achieve to associate and describe the observed regularity with additive and multiplicative relationships to express the quadratic behavior. This means, the pattern was expressed with a mathematical structure that comes from the functional relations between these values. In that sense, it is confirmed that determining a pattern involves the establishment of regularities and mathematical structures (Clements & Sarama, 2009).

The establishment of the pattern associated to the task is an essential process to go from the particular to the general. This fact agrees with the study on the inductive reasoning of pre-grade students reported by Haverty et al. (2000) that highlights the importance of the identification of patterns for obtaining formulas of functions based on numerical data. On our part, we shed light on how teachers connect the particular cases to establish the quadratic pattern. It can be said that most of the teachers used numerical strategies based on additive and multiplicative relationships instead of figural strategies for the analysis and description of the increasing behavior of the numerical data although the tasks were presented in a figural form. The different numerical relations established produced three types of structures (see **Table 1**) to express the pattern associated to the quadratic sequence in the task.

Table 1. General structure of the quadratic pattern established by the teachers

Structure of the quadratic pattern		Description
a)	$a_n = pn^2 + qn$	Additive relation of the square of the values of a variable and a multiple of these values
b)	$a_n = pn\left(n + \frac{q}{p}\right)$	Multiplicative relation between two linear factors of one variable
c)	$a_n = pn(n+1) + (q-p)n$	Juxtaposition of additive and multiplicative relations of one variable

Giving priority to the multiplicative or additive relations could be connected to the teachers' knowledge of numerical relationships (Glaser & Pellegrino, 1982; Haverty et al., 2000) or by their level of abstraction of the relationships involved in the structure of the pattern (Rivera, 2010). The focus on multiplicative and additive relationships between the values of each variable over the perception of the common characteristics between specific cases allowed to extend the pattern to a general class of elements, and therefore, to obtain a general rule. As a consequence, we conclude that taking out the context of the task and focusing on the study of the mathematical relationships that connect several particular cases promoted the transition from the establishment of a pattern to the formulation of the generalization because it was possible to abstract the present pattern in the data of the task.

Furthermore, we believe that using a generalization task of a figural pattern that is low in Gestalt goodness, where the pattern is not easily discernable in a visual way and has a more complex formulation of the generalization, contributes to the investigation of the mental processes underlying the inductive reasoning that lead to successful generalizations. This study also contributes, to some extent, to anticipate possible difficulties in more complex tasks of generalization. We consider that these types of results complement those reported in previous studies (Manfreda et al., 2012; Rivera & Becker, 2003) that used tasks of generalization of figural patterns that are high in Gestalt goodness.

One of the difficulties in the inductive reasoning of the teachers to determine the general rule associated to the quadratic pattern was the use of a recursive strategy to identify numerical regularities between the particular cases; this has been documented in several studies (e.g. Manfreda, et al., 2012; Rivera & Becker, 2007). In fact, this strategy made it harder to observe regularities that included several particular cases and to identify the relationship between them.

Another difficulty detected was the ability to associate the observed regularities with a mathematical structure that describes them. This difficulty is attributed to the lack of strategies to translate the numerical representation of the quadratic behavior to an algebraic expression of a polynomial function. Moreover, the abstraction from the particular to the general is a complex procedure for many teachers because they are not able to identify the norm that rules the behavior in the particular cases or the structure of the pattern. This could come from drawing too much attention to the context of the task instead of looking for the functional relationships between the variables.

Finally, and based on the results, it is considered that the three inductive processes described in this paper could be a reference to elaborate a much deeper investigation on the development of the inductive reasoning of mathematics teachers. Therefore, this study could initially contribute to overcome the difficulties of the teachers to obtain generalizations and could additionally become the guide for the emergence of a pedagogical framework oriented towards the improvement of the practice and development of inductive reasoning in class.

Disclosure statement

No potential conflict of interest was reported by the authors.

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