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Fractional Stochastic Heat Equation on the Half-Line

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Abstract

In this paper, we consider an initial-boundary value problem for a stochastic non-linear heat equation with Riemann-Liouville space-fractional derivative and white noise on the half-line. We construct the integral representation of the solution and prove existence and uniqueness. Moreover, we adapt stochastic integration methods to approximate the solutions

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1 Introduction

In the past years, fractional partial differential equations have been highly attractive to physicians, mathematicians, engineers, biologists, among others. This equations have a great applications to a wide range of phenomena (see [1, 3, 5, 8, 12, 13]). Space-fractional diffusion equations have been considered in the literature by numerous authors. For example, Vásquez et al., [18] studied a generalization of the second law of thermodinamics in the framework of the fractional calculus. Magin et al., [11] investigated the anomalous diffusion stretched exponential model which is used to detect and characterize neurodegenerative, malignant and ischemic diseases, and they incorporate a fractional order Brownian model diffusivity. Also, in recent surveys, the diffusion phenomena has been treated with stochastic partial differential equations (see [7, 10, 16]). In this cases, the authors prove the existence, uniqueness and regularity of mild solutions for the Cauchy problem.

In this paper, we consider an initial-boundary value problem for a stochastic non-linear heat equation

$$\begin{cases} q_t(x,t) = D^\alpha q(x,t) + \mathcal{N}q(x,t) + \dot{B}(x,t), & x>0, \quad t\in[0,T], \\ q(x,0) = q_0(x), \\ I^{2-\alpha}q(0,t) = g_2(t), \end{cases} \tag{1}$$

where D^α and I^α are the Riemann-Liouville fractional derivative and integral, respectively, $1<\alpha<2$, $\mathcal{N}$ is a lipschitzian operator and $\dot{B}(x,t)$ is the white noise. Let us notice that constructing the Green's function is not an easy task, due to the fact that the symbol of the differential operator is a multivalued function. We have successfully overcome this difficulty following the main ideas of the Fokas method [9]. Then, using the Green's function we construct the integral representation of the solution and prove existence and uniqueness, via Picard iteration scheme. Also, we adapt Monte Carlo integration methods to approximate the integral representation of the solution. In a previous work [2], we have considered a similar problem as (1), but with the Riesz fractional derivative of order $\alpha\in(2,3)$, where the symbol is an univalued function, instead of the Riemann-Liouville fractional derivative with $\alpha\in(1,2)$.

2 Linear problem

In this section, we consider the homogeneous linear problem associated to initial-boundary value problem (1),

$$\begin{cases} q_t(x,t) = D^\alpha q(x,t), & x>0, \quad t\in[0,T], \\ q(x,0) = q_0(x), \\ I^{2-\alpha}q(0,t) = g_2(t), \end{cases} \tag{2}$$

where $\alpha \in (1,2)$. The Riemann-Liouville fractional derivative is defined by the integral operator

$$D^\alpha f(x) = D^{1+[\alpha]} I^{1+[\alpha]-\alpha} f(x), \qquad I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt,$$

where $x > 0$, $\alpha > 0$, $[\alpha]$ is the integer part of α and Γ is the Gamma function (see, [15]). Now, we define the Fourier-Laplace transformation

$$\widehat{q}(k,t) = \int_0^\infty e^{-ikx} q(x,t) dx, \quad Im(k) < 0,$$

and its inverse by

$$q(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx} \widehat{q}(k,t) dk.$$

Applying the Fourier-Laplace transform to equation (2), we obtain

$$\widehat{q}_t(k,t) = (ik)^\alpha \widehat{q}(k,t) - g_1(t) - ikg_2(t),$$

where $(ik)^\alpha = |k|^\alpha e^{i\alpha(arg(ik))}$, $g_1(t) = D^{\alpha-1} q(0,t)$ and $g_2(t) = I^{2-\alpha} q(0,t)$. Here, we choose $arg(k)$ the following way: for $\alpha \in (1, 4/3]$, $\frac{\pi(2-3\alpha)}{\alpha} < arg(k) \leq \frac{\pi(2-\alpha)}{\alpha}$, and for $\alpha \in (4/3, 2)$, $-3\pi/2 < arg(k) \leq \pi/2$. Multiplying the above equation by $e^{-(ik)^\alpha t}$ and integrating with respect to the time variable we get, for $Im(k) < 0$,

$$e^{-(ik)^\alpha t} \widehat{q}(k,t) = \widehat{q_0}(k) - \widetilde{g}_1\left(-(ik)^\alpha, t\right) - ik\widetilde{g}_2\left(-(ik)^\alpha, t\right), \tag{3}$$

where

$$\widetilde{g}_j(k,t) = \int_0^t e^{ks} g_j(s) ds, \quad j = 1, 2.$$

Using the inverse Fourier-Laplace transform in (3) we find

$$\begin{aligned} q(x,t) = & \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx+(ik)^\alpha t} \widehat{q_0}(k) dk - \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx+(ik)^\alpha t} \widetilde{g}_1\left(-(ik)^\alpha, t\right) dk \\ & - \frac{i}{2\pi} \int_{-\infty}^{\infty} e^{ikx+(ik)^\alpha t} k\widetilde{g}_2\left(-(ik)^\alpha, t\right) dk. \end{aligned} \tag{4}$$

Now, we notice that if $arg(k) \in \left(\frac{-(3+\alpha)\pi}{2\alpha}, \frac{-(1+\alpha)\pi}{2\alpha}\right) \cup \left(\frac{(1-\alpha)\pi}{2\alpha}, \frac{(3-\alpha)\pi}{2\alpha}\right)$, then $Re(ik)^\alpha < 0$. Let's deform the contour of integration to ∂A in equation (4), where the region A is defined by $arg(k) \in (\frac{\pi(2-3\alpha)}{\alpha}, -(\pi+\epsilon))$, for $\alpha \in (1, 4/3]$ and $\epsilon > 0$ sufficient small, and $arg(k) \in (\frac{-3\pi}{2}, \frac{-2\pi}{\alpha}) \cup (\frac{\pi(2-\alpha)}{\alpha}, \frac{\pi}{2})$, for $\alpha \in (4/3, 2)$, thus

$$\begin{aligned} q(x,t) = & \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx+(ik)^\alpha t} \widehat{q_0}(k) dk - \frac{1}{2\pi} \int_{\partial A} e^{ikx+(ik)^\alpha t} \widetilde{g}_1\left(-(ik)^\alpha, t\right) dk \\ & - \frac{i}{2\pi} \int_{\partial A} e^{ikx+(ik)^\alpha t} k\widetilde{g}_2\left(-(ik)^\alpha, t\right) dk. \end{aligned} \tag{5}$$

We make the change of variable $k \to ke^{-i\frac{2\pi}{\alpha}}$ in (3) to obtain

$$e^{-(ik)^\alpha t}\widehat{q}\left(ke^{-i\frac{2\pi}{\alpha}},t\right) = \widehat{q_0}\left(ke^{-i\frac{2\pi}{\alpha}}\right) - \widetilde{g}_1\left(-(ik)^\alpha,t\right) - ike^{-i\frac{2\pi}{\alpha}}\widetilde{g}_2\left(-(ik)^\alpha,t\right). \tag{6}$$

Note that the above equation is valid for $Re(k) < \cot\left(\frac{2\pi}{\alpha}\right) Im(k)$. Also, by the Cauchy theorem

$$\int_{\partial A} e^{ikx}\widehat{q}\left(ke^{-i\frac{2\pi}{\alpha}},t\right)dk = 0.$$

Then, we substitute $\widetilde{g}_1\left(-(ik)^\alpha,t\right)$ from equation (6) in equation (5), thus using the above equation we arrive to

$$q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx+(ik)^\alpha t}\widehat{q_0}(k)dk - \frac{1}{2\pi}\int_{\partial A} e^{ikx+(ik)^\alpha t}\widehat{q_0}\left(ke^{-i\frac{2\pi}{\alpha}}\right)dk + \frac{e^{-i\frac{\pi}{\alpha}}\sin\left(\frac{\pi}{\alpha}\right)}{\pi}\int_{\partial A} e^{ikx+(ik)^\alpha t}k\widetilde{g}_2\left(-(ik)^\alpha,t\right)dk.$$

Therefore, we have obtained the integral representation

$$q(x,t) = \int_0^\infty G^{(I)}(x-y,t)q_0(y)dy + \int_0^t G^{(B)}(x,t-s)g_1(s)ds,$$

where the Green's function is given by

$$G^{(I)}(x,y,t) = \frac{1}{2\pi}\left(\int_{-\infty}^{\infty} e^{ik(x-y)+(ik)^\alpha t}dk - \int_{\partial A} e^{ik\left(x-ye^{-i\frac{2\pi}{\alpha}}\right)+(ik)^\alpha t}dk\right)$$

and

$$G^{(B)}(x,t-s) = \frac{e^{-i\frac{\pi}{\alpha}}\sin\left(\frac{\pi}{\alpha}\right)}{\pi}\int_{\partial A} ke^{ikx+(ik)^\alpha(t-s)}dk.$$

3 Existence and uniqueness of the solution

In this section, we prove the local existence and uniqueness of solutions to the stochastic initial and boundary value problem

$$\begin{cases} q_t(x,t) = D^\alpha q(x,t) + \mathcal{N}q(x,t) + \dot{B}(x,t), & x>0, \quad t\in[0,T], \\ q(x,0) = q_0(x), \\ I^{2-\alpha}q(0,t) = g_2(t), \end{cases} \tag{7}$$

where $\alpha \in (1,2)$, $\mathcal{N}$ is a lipschitzian operator and $\dot{B}(x,t)$ is the white noise on $\mathbb{R}^+ \times [0,T]$, defined on a complete probability space $(\Omega,\mathcal{F},\mathcal{F}_t,P)$, where P is a probability measure, $\mathcal{F}$ is a σ-algebra and $\{\mathcal{F}_t\}_{t\geq 0}$ is a right-continuous filtration on $(\Omega, \mathcal{F})$ such that $\mathcal{F}_0$ contains all P-negligible subsets. Let the set $B = \{B(x,t)|x \in \mathbb{R}^+, t \geq 0\}$ be a centered Gaussian field with covariance function given by

$$K((x,t),(y,s)) = \min\{x,y\}\min\{t,s\}.$$

We suppose that B generates a $(\mathcal{F}_t, t \geq 0)$-martingale measure in the sense of Walsh [19]. The initial condition q_0 is supposed to be $\mathcal{F}_0 \times \mathcal{B}(\mathbb{R}^+)$ measurable, where $\mathcal{B}(\mathbb{R}^+)$ is the Borelian σ-algebra over $\mathbb{R}^+$.

We understand the equation (7) in the Walsh [19] sense. That is, q is called a solution if for all $x \in \mathbb{R}^+$ and $t \in [0,T]$, q satisfies

$$\begin{aligned} q(x,t) &= \int_0^\infty G^{(I)}(x,y,t)q_0(y)\,dy + \int_0^t G^{(B)}(x,t-s)g_1(s)\,ds \\ &\quad + \int_0^t\int_0^\infty G(x-y,t-s)\mathcal{N}q(y,s)\,dyds \\ &\quad + \int_0^t\int_0^\infty G(x-y,t-s)\,dB(y,s) \end{aligned} \tag{8}$$

where

$$G(\xi,\tau) = \frac{1}{2\pi}\int_{-\infty}^\infty e^{ik\xi+(ik)^\alpha\tau}\,dk. \tag{9}$$

The field $\{q(x,t)|x \in \mathbb{R}^+, t \geq 0\}$ is said to be a global mild solution of equation (8) if, for all $0 < T < \infty$, $\{q(x,t)|x \in \mathbb{R}^+, t \in [0,T]\}$ is a mild solution on the interval $[0,T]$. Furthermore, a global mild solution is in $L^p(\Omega)$ for some $p \geq 1$ if, for all $T \geq 0$, $x \in \mathbb{R}^+$, and for all $t \in [0,T]$, $\sup\{\mathbb{E}(|q(x,t)|^p)|(x,t) \in \mathbb{R}^+ \times [0,T]\} < \infty$, where $\mathbb{E}$ is the expectation with respect to P. We enunciate the Gronwall's Lemma, which is going to be used in the proof of Theorem 3.2,

Lemma 3.1. *Suppose $\phi_1, \phi_2, \ldots : [0,T] \to \mathbb{R}^+$ are measurable and non-decreasing. Suppose also that there exist a constant A such that for all integers $n \geq 1$, and $t \in [0,T]$,*

$$\phi_{n+1}(t) \leq A\int_0^t \phi_n(s)\,ds.$$

Then,

$$\phi_n(t) \leq \phi_1(t)\frac{(At)^{n-1}}{(n-1)!}.$$

Now, we prove the existence and uniqueness theorem, where we understand the uniqueness of the solution in (8) in the sence that for any mild solutions q_1 and q_2 on $[0,T]$ we have $q_1(x,t) = q_2(x,t)$ in $L^p(\Omega)$, $p \geq 1$, for all $x \in \mathbb{R}^+$ and for all $t \in [0,T]$.

Theorem 3.2. *Suppose that for each $T > 0$, there exists a constant $C > 0$ such that for each $(x,t) \in \mathbb{R}^+ \times [0,T]$,*

$$|\mathcal{N}q_1 - \mathcal{N}q_2| \leq C|q_1 - q_2|,$$

and for some $p \geq 1$,

$$\sup_{x\in\mathbb{R}^+} \mathbb{E}(|q_0(x)|^p) < \infty. \tag{10}$$

Then, there exists a unique solution $q(x,t)$ to equation (7). Moreover, for all $T > 0$ and $p \geq 1$,

$$\sup_{(x,t)\in\mathbb{R}^+\times[0,T]} \mathbb{E}(|q(x,t)|^p) < \infty.$$

Proof. Let's define

$$\begin{aligned} q^{n+1}(x,t) &= q^0(x,t) + \int_0^t G^{(B)}(x,t-s)g_1(s)\,ds \\ &\quad + \int_0^t \int_0^\infty G(x-y,t-s)\mathcal{N}q^n(y,s)\,dyds \qquad (11) \\ &\quad + \int_0^t \int_0^\infty G(x-y,t-s)\,dB(y,s), \end{aligned}$$

where

$$q^0(x,t) = \int_0^\infty G^{(I)}(x,y,t)q_0(y)dy.$$

First, we are going to prove that $\{q^n(x,t)\}_{n\geq 0}$ converges in $L^p(\Omega)$. As for $n \geq 2$,

$$\begin{aligned} &\mathbb{E}(|q^{n+1}(x,t) - q^n(x,t)|^p) \\ &= \mathbb{E}\left(\left|\int_0^t \int_0^\infty G(x-y,t-s)[\mathcal{N}q^n(y,s) - \mathcal{N}q^{n-1}(y,s)]\,dyds\right|^p\right) \\ &\leq C(p)\int_0^t \int_0^\infty G(x-y,t-s)\mathbb{E}(|q^n(y,s) - q^{n-1}(y,s)|^p)dyds \\ &\leq C(p)\int_0^t \sup_{y\in\mathbb{R}^+} \mathbb{E}(|q^n(y,s) - q^{n-1}(y,s)|^p)ds \end{aligned}$$

and by (10),

$$\begin{aligned} &\sup_{x\in\mathbb{R}^+} \mathbb{E}(|q^1(x,t) - q^0(x,t)|^p) \\ \leq\ & C(p)\left(\sup_{x\in\mathbb{R}^+} \mathbb{E}(|q^1(x,t)|^p) + \sup_{x\in\mathbb{R}^+} \mathbb{E}(|q^0(x,t)|^p)\right) < \infty. \end{aligned}$$

Then, Lemma 3.1 shows that

$$\sum_{n\geq 0} \sup_{(x,t)\in\mathbb{R}^+\times[0,T]} \mathbb{E}(|q^n(x,t) - q^{n-1}(x,t)|^p) < \infty.$$

Hence, $\{q^n(x,t)\}_{n\geq 0}$ is a Cauchy sequence in $L^p(\Omega)$. Let

$$q(x,t) = \lim_{n\to\infty} q^n(x,t).$$

Then for each $(x,t) \in \mathbb{R}^+ \times [0,T]$,

$$\sup_{(x,t)\in\mathbb{R}^+\times[0,T]} \mathbb{E}(|q(x,t)|^p) < \infty.$$

Take $n \to \infty$ in $L^p(\Omega)$ at both sides of (11). Then, it shows that $q(x,t)$, satisfies the problem (7). Finally, we have to prove the uniqueness of the solution. Let q_1 and q_2 be the two solutions of problem (7), then

$$\begin{aligned}
&\mathbb{E}(|q_1(x,t) - q_2(x,t)|^p) \\
&= \mathbb{E}\left(\left|\int_0^t \int_0^\infty G(x-y,t-s)[\mathcal{N}q_1(y,s)) - \mathcal{N}q_2(y,s))]dyds\right|^p\right) \\
&\leq C(p)\int_0^t \int_{\mathbb{R}} G(x-y,t-s)\mathbb{E}(|q_1(y,s) - q_2(y,s)|^p)dyds \\
&\qquad \leq C(p)\int_0^t \sup_{y\in\mathbb{R}^+} \mathbb{E}(|q_1(y,s) - q_2(y,s)|^p)ds.
\end{aligned}$$

For Lemma 3.1 we obtain

$$\mathbb{E}(|q_1(x,t) - q_2(x,t)|^p) = 0.$$

The Theorem 3.2 is proved.

□

4 Numerical solution

In this section, we implement Monte Carlo methods to give an approximation $q_{MC}(x,t)$ to the solution $q(x,t)$ of the stochastic heat equation (8). By the Law of Large Numbers, $q_{MC}(x,t)$ converges to $q(x,t)$ in $L^p(\Omega)$ [17]. This methods have proved to be efficient due to their simplicity and accuracy, even with limited computing power. In this paper, we use such computational tools to reach our target (see [4, 6, 17]). We apply a numerical strategy for the multiple integrals, implementing Importance Samplig algorithms whitin direct Monte Carlo integration methods.

In order to implement Monte Carlo integration, note that the real part of equation (9) is

$$G(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{|k|^{\alpha}\cos(arg(ik)^{\alpha})t}\cos(kx + |k|^{\alpha}\sin(arg(ik)^{\alpha})t)\,dk. \tag{12}$$

We briefly describe the procedure that we conducted to get the numerical approximation of the solution in (8). For nth Picard iteration, we compute the next steps:

1. For the first integral in (8) with the initial condition, we choose as the initial condition q_0, the Uniform probability density function (pdf), with support on $(3,4)$, in order to apply Monte Carlo integration and Important Sampling method for the Green function.

2. For the integral with the non-linear operator, we used Important Sampling method for all integrals, with important functions: Uniform pdf on $[0,t]$ and bivariate Exponential pdf for space.

3. For the stochastic integral, we used composition method to obtain its realizations.

For this numerical example we take $g_2(t) = 0$, for all $t \in [0,T]$, $\alpha = 3/2$ and $n = 10$ Picard iterations were calculated. R statistical software [14] has been used to carry out this procedure. For each (x,t) we calculated the estimated Monte Carlo Standard Error MCSE of $q_{MC}(x,t)$. In the Figure 1 we show the MCSE for the case $x = 3$, $t = 1$, and we observe that samples of size 15,000 are enough to reach satisfying results. In the Figure 2, we show the numerical representation of the solution.

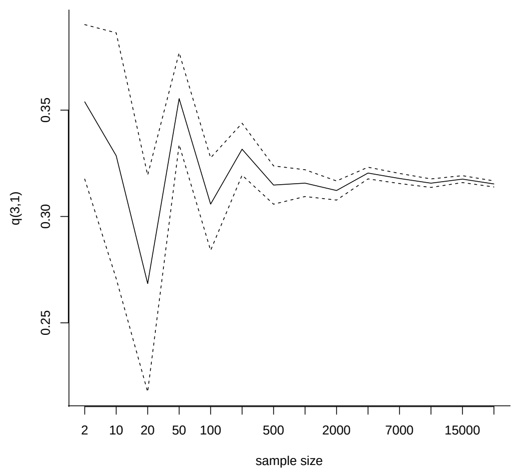

Figure 1: Estimated Monte Carlo Standard Error

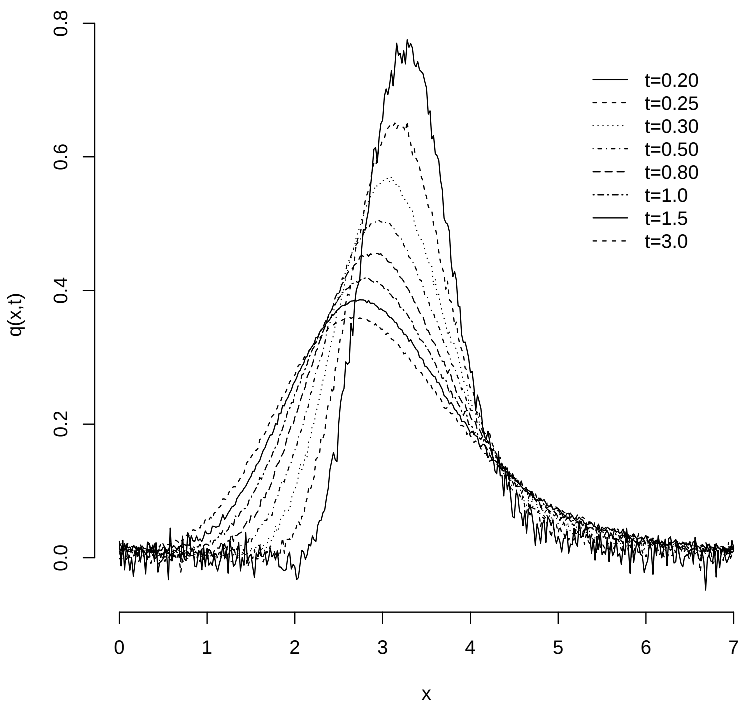

Figure 2: Numerical aproximation

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